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THE STIFFNESS PROBLEM FOR STOCHASTIC SYSTEMS AND A METHOD OF SOLVING IT†

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Using exact or approximate first integrals of the so-called truncated system of differential equations, it is shown that it is possible to improve the stability of the numerical analysis of dynamical systems described by stiff stochastic differential equations. A numerical example is given. © 1999 Elsevier Science Ltd. All rights reserved.

The analysis and synthesis of dynamical systems which operate under random perturbations often involves the integration of stochastic differential equations (SDE). There are a number of specific problems, such as filtering, identification, prediction and optimal control, for which the SDE must be solved in real time. However, the computations involved are often difficult, due to the high dimension of the equations [1, 2]. This is even more evident in the case of SDE which exhibit the stiffness effect [3–8], which is expressed in the presence of boundary layers.

Stiffness is an internal property of a dynamical system whose description at any point in the observation segment requires functions of two forms: one which is rapidly decaying and has large derivatives and one with small derivatives.

Stiff dynamical systems have to be considered separately because of the difficulty of using classical methods, such as Adams and Runge–Kutta integration, to solve the SDE. We know [3, 5–8] that the small integration step used to reproduce fast processes in a boundary layer cannot be increased outside the layer, even though the derivatives are much smaller there. Even a very small increase in the step can lead to rapid increase ("an explosion") in the error. There is a conflict between having a large enough interval for interpolation of the solution and an acceptable integration step.

A number of ways in which it is possible to increase the integration step outside the boundary layer are described in [3–8]. However, there has not been wide use of implicit methods, which are the ones mainly considered in those studies, in the theory and practice of optimal filtering, identification, prediction and control.

The stiffness effect can be removed by using the first integrals of the so-called truncated system of differential equations described below. This improves the stability of the numerical methods employed while allowing a large increase in the integration step.

1. BASIC DEFINITIONS. STATEMENT OF THE PROBLEM

Suppose the vector of state $X^T(t) = [x_i(t)]$ of the dynamical system is described by a system of SDE in symmetricized form

$$d\mathbf{X}/dt = \mathbf{A}(t,\mathbf{X}) + G(t)\mathbf{n}(t), \quad \mathbf{X}(t_0) = \mathbf{X}_0 \tag{1.1}$$

where $\mathbf{A}^T(t, \mathbf{X}) = [a_i(t, x)]$, $G(t) = [g_{ij}(t)]$ are known matrices with elements $a_i(t, x)$ and $g_{ij}(t)$, which are continuously differentiable functions satisfying the Lipschitz condition, $\mathbf{n}^T(t) = [n_i(t)]$ is the vector of white Gaussian noise with given statistical characteristics: $\langle\mathbf{n}(t)\rangle = 0\langle\mathbf{n}(t)\mathbf{n}^T(t + \tau)\rangle = U\delta(\tau)$, $U = [u_{ij}]$ is a symmetric non-negative definite matrix of the intensity of the white noise, T denotes the transpose and $\langle\cdot\rangle$ denotes the mathematical expectation. Here and henceforth, unless otherwise stated, $t \in [t_0, t_0 + T]$ and the subscripts i and j take the values $1, 2, \ldots, N$.

We know [3] that the stiffness of a system of SDE of the form (1.1) is determined by the behaviour of the solution of the truncated deterministic system

$$d\overline{X}/dt = \mathbf{A}(t,\overline{\mathbf{X}}), \quad \overline{\mathbf{X}}(t_0) = \overline{\mathbf{X}}_0 = \mathbf{X}_0 \tag{1.2}$$

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Definition 1 [8]. A system of SDE of the form (1.1) is said to be stiff if the spectrum of the matrix $[\partial \mathbf{A}(t, \bar{\mathbf{X}})/\partial \bar{\mathbf{X}}]$ can be clearly divided into two parts (assuming $N = K + M$): a stiff spectrum whose eigenvalues satisfy the conditions

$$\operatorname{Re}\overset{*}{\lambda}_k \leqslant -L, \ |\operatorname{Im}\overset{*}{\lambda}_k| < |\operatorname{Re}\overset{*}{\lambda}_k|, \ k = 1,2,\ldots,K$$

and a soft spectrum for which

$$|\tilde{\lambda}_m| \leqslant l \ll L, \quad m = 1,2,\ldots,M$$

The ratio L/l
is called the stiffness index of the system.
The following definition is also used in practice.

Definition 2 [3]. The system of SDE (1.1) is said to be stiff if

$$\begin{aligned} &|\lambda_i| \exp(\operatorname{Re}\lambda_i \tau_{ps}) \leqslant L_{ps}/N_{ps}, \quad \operatorname{Re}\lambda_i < 0; \\ &|\lambda_i| \leqslant L_{ps}/N_{ps}, \quad \operatorname{Re}\lambda_i \geqslant 0 \\ &L_{ps} = \max|\lambda_i|, \quad N_{ps} \gg 1, \quad \tau_{ps} \ll T \end{aligned} \tag{1.3}$$

Corollary. It follows at once that a stiff system cannot have eigenvalues with a large modulus (of order L_{ps}) which have a positive real part. For eigenvalues whose modulus is of order L_{ps}, we must have $\exp(\operatorname{Re}\lambda_i \tau_{ps}) \leqslant N_{ps}^{-1}$, $N_{ps} \gg 1$, that is, they must have large negative real parts.
We could also find out whether system (1.1) is stiff or not by integrating it on the initial segment by Euler's method. Any attempt to increase the small discreteness step $\Delta\tau$ $(\Delta\tau \leqslant \|\partial \mathbf{A}(t, \bar{\mathbf{X}})/\partial \bar{\mathbf{X}}\|^{-1})$ leads to an exponential increase in the error.

It is worth noting that the stiffness criterion can be changed, depending on the form of the SDE. Consider, for example, a SDE of the form

$$d\mathbf{X}/dt = \mathbf{A}(t,\mathbf{X}) + \sum_{m=1}^{M} \mathbf{G}_m(t,\mathbf{X}) n_m(t), \quad \mathbf{X}(t_0) = \mathbf{X}_0 \tag{1.4}$$

where $\mathbf{A}(t, \mathbf{X}) = [a_i(t, X)]^T$, $\mathbf{G}_j(t, x) = [g_{ij}(t, X)]^T$ are known vector functions and $n_1(t), n_2(t), \ldots, n_M(t)$ represent white Gaussian noise with given statistical parameters and are independent of one another.
In this case the criteria involve analysing the eigenvalues of a matrix of the form [4]

$$\begin{aligned} &\hat{H}(t) = H(t) + H^T(t) + \sum_{m=1}^{M} B_m(t) B_m^T(t) \\ &H(t) = \partial \mathbf{A}(t,\mathbf{X})/\partial \mathbf{X}, \quad B_m(t) = \partial \mathbf{G}_m(t,\mathbf{X})/\partial \mathbf{X}, \quad m = 1,2,\ldots,M \end{aligned} \tag{1.5}$$

As a rule, it is because SDE (1.1) and (1.4) are stiff that the numerical methods used to integrate them are unstable. Obviously, if we could use a sufficiently small integration step, the methods would be stable, but the demand on computational resources would become unreasonable.

2. TRANSFORMATION OF THE STOCHASTIC DIFFERENTIAL EQUATIONS

Suppose system (1.2) has a known exact or approximate analytical solution

$$\bar{\mathbf{X}}(t) = \mathbf{F}(t, \bar{\mathbf{X}}_0), \quad \bar{\mathbf{X}}(t_0) = \bar{\mathbf{X}}_0 = \mathbf{X}_0 \tag{2.1}$$

An approximate analytical solution can be obtained by the method of supporting integral curves [9–12], in which

$$\bar{\mathbf{X}}(t) = C(\bar{\mathbf{X}}_0)\mathbf{V}(t)$$

where $C(\bar{X}_0) = [c_{ij0}(\bar{X}_0)]$ $(j_0 = 1, 2, \ldots, N_0)$ is a matrix of known coefficients which depend continuously on the initial condition, $\bar{\mathbf{X}}_0$, and $\mathbf{V}(t) = [v_1(t), v_2(t), \ldots, v_{N0}(t)]$ is a vector of independent functions.

By analogy with the approach used previously in [13], for (1.2), allowing for (2.1), we define the vector of independent first integrals

$$\mathbf{W}(t, \bar{\mathbf{X}}, \bar{\mathbf{X}}_0) = \bar{\mathbf{X}} - \mathbf{F}(t, \bar{\mathbf{X}}_0) \tag{2.2}$$

$\mathbf{W}(t, \bar{\mathbf{X}}, \bar{\mathbf{X}}_0) = [w_i(t, \bar{\mathbf{X}}, \bar{\mathbf{X}}_0)]^T$, $d\bar{\mathbf{X}}_0/dt \equiv 0$, which satisfies the condition

$$\frac{\partial}{\partial t}\mathbf{W}(t, \bar{\mathbf{X}}, \bar{\mathbf{X}}_0) + \left\{\mathbf{A}^T(t, \bar{\mathbf{X}}) D_X[\mathbf{W}(t, \bar{\mathbf{X}}, \bar{\mathbf{X}}_0)]\right\}^T = 0 \tag{2.3}$$

and, on any solution $\bar{\mathbf{X}}(t)$ of system (1.2), becomes the identity

$$\mathbf{W}(t, \bar{\mathbf{X}}(t), \bar{\mathbf{X}}_0) = \bar{\mathbf{X}}(t) - \mathbf{F}(t, \bar{\mathbf{X}}_0) \equiv 0, \tag{2.4}$$

$$D_{\bar{X}}[\mathbf{W}(t, \bar{\mathbf{X}}, \bar{\mathbf{X}}_0)] = [\partial w_i / \partial \bar{x}_j]^T$$

The expression $J\{\mathbf{W}(t, \bar{\mathbf{X}}, \bar{\mathbf{X}}_0)\}$ (where $J\{\cdot\}$ is any differentiable function) is obviously also a first integral of system (1.2).

If, instead of $\bar{\mathbf{X}}(t)$, we substitute the solution $\mathbf{X}(t)$ satisfying the SDE (1.1) into (2.4), we must replace the vector $\bar{\mathbf{X}}_0$ by $\mathbf{Q}(t) = [q_i(t)]^T$, for which

$$\mathbf{W}(t, \mathbf{X}(t), \mathbf{Q}(t)) = \mathbf{X}(t) - \mathbf{F}(t, \mathbf{Q}(t)) = 0, \quad d\mathbf{Q}/dt \not\equiv 0 \tag{2.5}$$

Differentiating the left- and right-hand sides of (2.5) with respect to t, taking (1.1) into account and noting that, by virtue of (2.3)

$$\frac{\partial}{\partial t}\mathbf{W}(t, \mathbf{X}, \mathbf{Q}) + \left\{\mathbf{A}^T(t, \mathbf{X}) D_X[\mathbf{W}(t, \mathbf{X}, \mathbf{Q})]\right\}^T = 0$$

we obtain

$$D_X^T[\mathbf{W}(t, \mathbf{X}, \mathbf{Q})] G(t)\mathbf{n}(t)_0 + D_Q^T[\mathbf{W}(t, \mathbf{X}, \mathbf{Q})]\frac{d\mathbf{Q}}{dt} = 0 \tag{2.6}$$

$$D_X[\mathbf{W}(t, \mathbf{X}, \mathbf{Q})] = [\partial w_i / \partial x_j]^T, \quad D_Q[\mathbf{W}(t, \mathbf{X}, \mathbf{Q})] = [\partial w_i / \partial q_j]^T$$

From (2.5) and (2.6) we have

$$G(t)\mathbf{n}(t) - D_Q^T[\mathbf{F}(t, \mathbf{Q})]\frac{d\mathbf{Q}}{dt} = 0$$

Hence

$$\frac{d\mathbf{Q}}{dt} = \{D_Q^T[\mathbf{F}(t, \mathbf{Q})]\}^{-1} G(t)\mathbf{n}(t) = \tilde{G}(t, \mathbf{Q})\mathbf{n}(t) = \sum_{j=1}^{N} \tilde{G}_j(t, \mathbf{Q}) n_j(t), \quad \tilde{G}_j = [\tilde{g}_{ij}]^T \tag{2.7}$$

It has been assumed here that the condition $\det D_Q^T[\mathbf{F}(t, \mathbf{Q})] \neq 0$ ($t \in [t_0, t_0 + T]$) is satisfied, thus ensuring that the inverse matrix $\{D_Q^T[\mathbf{F}(t, \mathbf{Q})]\}^{-1}$ exists.

Theorem. The system of SDE of the form (2.7) is not stiff.

Proof. Since system (2.7) is similar to (1.4) in form, we can evaluate its stiffness by analysing the eigenvalues of the matrix (1.5). As it applies to (2.7) we have

$$\tilde{H}(t) = \sum_{j=1}^{N} \tilde{B}_j(t)\tilde{B}_j^T(t), \quad \tilde{B}_j(t) = \partial\tilde{\mathbf{G}}_j / \partial\mathbf{Q} \tag{2.8}$$

A matrix of form (2.8) is symmetric (real Hermitian) and positive definite, and its eigenvalues are therefore real and positive [14]. Thus the conditions for stiffness stipulated by Definitions 1 and 2 and the corollary are not satisfied and a SDE of the form (2.7) is not stiff.

Using the above method we therefore transformed the initial stiff stochastic system (1.1) into the new non-stiff system (2.7). This enables us to relax the constraints on the choice of integration step in existing methods of solving differential equations, making it easier to obtain the required estimate in problems of filtering, identification, prediction and control.

The initial system (1.1) is solved using (2.7) as follows:

$$\mathbf{X}(t) = \mathbf{F}(t, \mathbf{Q}(t))$$

As we have seen, the first integrals required for the non-linear model (1.1) can be found by approximate analytic methods, including the method of supporting integral curves [9–12].

The above method is most simply and clearly applied to linear SDE. In that case we can write formula (2.1) as

$$\overline{\mathbf{X}}(t) = \Phi(t, t_0)\overline{\mathbf{X}}_0$$

where $\Phi(y, t_0)$ is the fundamental matrix of solutions of the homogeneous system (1.2). Then (2.7) will take the form

$$d\mathbf{Q}/dt = \Phi^{-1}(t, t_0)G(t)\mathbf{n}(t) \tag{2.9}$$

The computational effectiveness of the method is greatest in the case when the solution $\mathbf{X}(t)$ of Eq. (1.1) does not need to be known at each integration step $t_k \in [t_0, t_0 + T]$ $(k = 1, 2, \ldots, K)$ of Eq. (2.7), but only at certain intermediate times t_l $(l \in \{1, 2, \ldots, K\})$ or at a finite time $t_K = t_0 + T$.

3. EXAMPLE

We will apply the method to the problem of suboptimal estimation of the parameters of motion of an aircraft. In the time interval $[t_0, t_0 + T]$, suppose we observe a sample of the random process

$$\xi(t) = S(t, \mathbf{X}) + h(t) \tag{3.1}$$

where $h(t)$ is broadband fluctuation interference approximating white Gaussian noise with parameters: $\langle h(t)\rangle = 0$, $\langle h(t)h(t+\tau)\rangle = V\delta(\tau)$.

The useful signal in (3.1) is given by the expression [2]

$$S(t, \mathbf{X}) = A_0(t)\cos[\omega_0(t - 2c^{-1}D_r(t)) + \varphi(t)] \tag{3.2}$$

where $A_0(t)$ and ω_0 are the amplitude and carrier frequency of the useful signal, $D_r(t)$ is the inclined distance from the aircraft to the point of reflection of the signal and $\varphi(t)$ is the random phase of the signal.

We will use the following mathematical model [2] to characterize the motion of the aircraft and phase fluctuations of the useful signal

$$\frac{dD_r}{dt} = W_r, \quad \frac{dW_r}{dt} = a_r, \quad \frac{da_r}{dt} = -\alpha a_r + \sqrt{2\alpha\sigma_a^2}\,n_a, \quad \frac{d\varphi}{dt} = \Delta\omega + \sqrt{\frac{N_\varphi}{2n_\varphi}}$$
$$\frac{d\Delta\omega}{dt} = -\gamma_\omega\Delta\omega + \sqrt{2\gamma_\omega\sigma_\omega^2}\,n_\omega \tag{3.3}$$

where W_r is the projection of the ground speed of the aircraft in the direction of the speedometer beam, a_r is a random process defining the radial component of acceleration of the aircraft, $\Delta\omega$ is the uncompensated Doppler frequency shift and γ_ω and α are parameters characterizing the width of the Doppler frequency spectrum and the spectrum of fluctuations of the quantity a_r respectively.

Thus in this case the vector of state $\mathbf{X}(t)$ has the form

$$\mathbf{X}^T = [x_1 = D_r, x_2 = W_r, x_3 = a_r, x_4 = \varphi, x_5 = \Delta\omega]$$

The vector of state $\mathbf{x}(t)$ to be evaluated satisfies the *a priori* equation

$$d\mathbf{X}/dt = A\mathbf{X} + G\mathbf{N}, \quad \mathbf{X}(t_0) = \mathbf{X}_0 \tag{3.4}$$

where $\mathbf{N}^T(t) = [0, 0, n_a(t), n_\varphi(t), n_\omega(t)]$ is the vector of the white Gaussian noise with zero mathematical expectation and unit intensity and $A = [a_{ij}]$, $G = [g_{ij}]$ are matrices with non-zero elements: $a_{12} = a_{23} = a_{45} = 1$, $a_{33} = -\alpha$, $a_{55} = -\gamma_\omega$, $g_{33} = \sqrt{(2\alpha\sigma_a^2)}$, $g_{44} = \sqrt{(N_\varphi/2)}$, $g_{55} = \sqrt{(2\gamma_\omega\sigma_\omega^2)}$. In the given example, $N = 5$, that is, $i, j = 1, 2, \ldots, 5$.

After appropriate transformations the equation which determines the algorithm of quasi-optimal evaluation of the vector applied to (3.1), (3.2) and (3.4) becomes

$$d\hat{\mathbf{X}}/dt = A(t)\hat{\mathbf{X}}(t) + K(t)\mathbf{F}_1(t,\hat{\mathbf{X}}),\ \ \hat{\mathbf{X}}(t_0) = \mathbf{X}_0 \tag{3.5}$$

$$\mathbf{F}_1^T(t,\mathbf{X}) = [4\omega_0 c^{-1}V^{-1}A_0(t)\xi(t)\sin(\hat{\Phi}_0),\ 0,\ 0,\ -2V^{-1}A_0(t)\xi(t)\sin(\hat{\Phi}_0),\ 0]$$

$$\hat{\Phi}_0 = [\omega_0(t - 2\hat{D}_r(t)c^{-1}) + \hat{\varphi}(t)]$$

where $K(t) = [k_{ij}]$ is the covariation matrix of the filtering errors.

The potential accuracy and noise immunity characteristics of the optimal receiver and signal processor (3.2) can be obtained by solving the equation for the covariation matrix [2]. In this case it takes the form

$$dK/dt = AK + KA^T + P + KRK,\ \ P = GIG^T \tag{3.6}$$

The non-zero elements of the matrices $P = [p_{ij}]$ and $R = [r_{ij}]$ are given by the expressions

$$p_{33} = 2\alpha\sigma_a^2,\ \ p_{44} = \gamma_\omega D_\varphi,\ \ p_{55} = 2\gamma_\omega\sigma_\omega^2,\ \ r_{11} = -(8\omega_0^2/c^2)V^{-1}A_0^2$$

$$r_{14} = r_{41} = (4\omega_0/c)V^{-1}A_0^2,\ \ r_{44} = -2V^{-1}A_0^2$$

Using criterion (1.3) to evaluate the stiffness of the initial system (3.4) (allowing or the fact that $\tau_{ps} = 10^{-2}$ s), we determine the eigenvalues of the matrix $A(t)$: $\lambda_1 = 0$, $\lambda_2 = 0$, $\lambda_3 = 5\cdot10^{-2}$, $\lambda_4 = 0$, $\lambda_5 = -100$. By (1.3), for the values λ_3 and λ_5 respectively we have $N_{ps} = 2010$, $N_{ps} = 22026$, which means that system (3.4) is stiff (since $N_{ps} \gg 1$).

Moreover, the spectrum of the matrix $A(t)$ can be divided quite clearly into a stiff and a soft spectrum [8], with $N = 5, K = 1, M = 4, \lambda^* = \lambda_5, \lambda_2 = \lambda_2, \lambda_3 = \lambda_3, \lambda_4 = \lambda_4$. It is clear from the stiffness index, defined as $L/l = 2000$ (by Definition 1), for instance, that the problem is stiff.

According to earlier results [3], the step $\Delta\tau$, which ensures stable integration of systems (3.5), (3.6), is found from the condition $\Delta\tau \leqslant \| A(t) \|^{-1}$.

We will eliminate the stiffness by the method described above. Using (2.9), we find the fundamental matrix of solutions $\Phi(t, t_0)$ for a truncated system of the form (1.2).

The non-zero elements of the matrix $\Phi(t, t_0) = [\varphi_{ij}]$ are

$$\varphi_{11} = \varphi_{22} = \varphi_{44} = 1,\ \ \varphi_{12} = t - t_0,\ \ \varphi_{13} = \alpha^{-2}\{\alpha(t - t_0) - 1 + \xi_\alpha\}$$

$$\varphi_{23} = \alpha^{-1}\{1 - \xi_\alpha\},\ \ \varphi_{33} = \xi_\alpha,\ \ \varphi_{45} = \gamma^{-1}\{1 - \xi_\gamma\},\ \ \varphi_{55} = \xi_\gamma$$

$$(\xi_\alpha = \exp[\alpha(t - t_0)],\ \ \xi_\gamma = \exp[\gamma(t - t_0)])$$

The elements of the matrix $\Phi^{-1}(t, t_0) = \Phi_1(t, t_0)$ are found similarly, except that

$$\varphi_{(1)12} = -\varphi_{12},\ \ \varphi_{(1)13} = \alpha^{-2}\{\xi_\alpha - \alpha(t - t_0) - 1\}$$

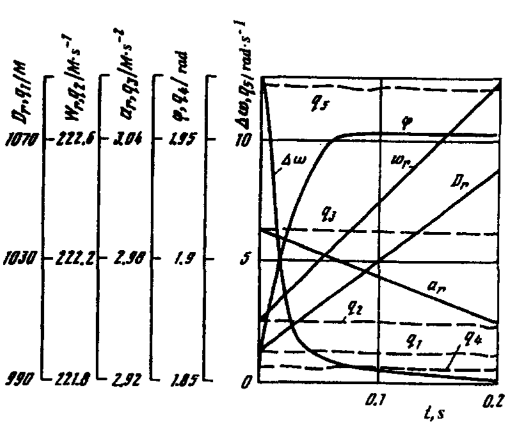

Fig. 1.

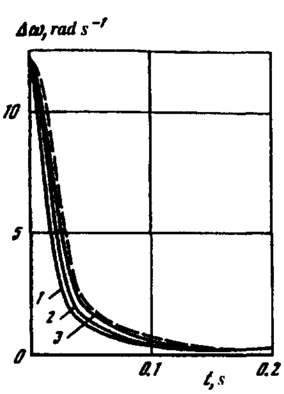

Fig. 2.

In this case, the non-stiff algorithm of quasi-optimal evaluation (3.5), (3.6) takes the form

$$d\hat{\mathbf{Q}}/dt = \tilde{K}\mathbf{F}_1(t,\hat{Q}), \quad \hat{\mathbf{Q}}(t_0) = \hat{\mathbf{X}}(t_0) \tag{3.7}$$

$$d\tilde{K}/dt = \tilde{P} + \tilde{K}R\tilde{K}, \quad \tilde{P} = \tilde{G}I\tilde{G}^T, \quad \tilde{K}(t_0) = K(t_0)$$

where $\tilde{G} = \Phi^{-1}(t, t_0)$.

The transition from $\hat{\mathbf{Q}}$ to $\hat{\mathbf{X}}$ is made by the formula

$$\hat{\mathbf{X}}(t) = \Phi(t, t_0)\hat{\mathbf{Q}}(t) \tag{3.8}$$

The non-zero elements of the vector $F_1(t, \hat{Q}) = [f_{(1)i}]^T$ are given by the expressions

$$f_{(1)1} = 4\omega_0 c^{-1}V^{-1}A_0(t)\xi(t)\sin(\hat{\Phi}_2), \quad f_{(1)4} = 2V^{-1}A_0(t)\xi(t)\sin(\hat{\Phi}_2)$$

$$\hat{\Phi}_2 = \omega_0\{t - 2c^{-1}[\hat{q}_1(t) + \hat{q}_2(t)(t-t_0) + \alpha^{-2}[\alpha(t-t_0) - 1 + \xi_\alpha^{-1}]\hat{q}_3(t)] + \gamma^{-1}[1 - \xi_\gamma^{-1}]\hat{q}_4(t)\}$$

The quantity $A_0^2/(2\gamma V)$ characterizes the signal-to-noise ratio in the observed sample $\xi(t)$.

A computer simulation was carried out with the following initial data: $\gamma_\omega = 10^2\ \mathrm{s}^{-1}$, $\alpha = 5 \times 10^{-2}\ \mathrm{s}^{-1}$, $\sigma_D = 10^2$ m, $\sigma_W = 30\ \mathrm{ms}^{-1}$, $\sigma_a = 10\ \mathrm{ms}^{-2}$, $\sigma_\omega = 1$–10 Hz, $\sigma_\varphi = \pi\sqrt{3}$.

The results are shown in Fig. 1. The solid lines show the estimates for the vector of state $X(t)$ obtained by integrating the quasi-optimal system (3.5), (3.6) by a fourth-order Runge–Kutta method with step $\Delta\tau = 10^{-3}$. This is then taken as exact. From the shape of the curve for $\Delta\omega$ we see that system (3.4) is stiff. The dashed line shows the result of integrating system (3.7), after using the above method to compensate for stiffness in the stochastic system (3.4). By comparing the estimates for $X(t)$ and $Q(t)$ we can see that the stiffness is eliminated as a result of the transformation.

We evaluated the computational efficiency of the quasi-optimal estimation algorithm (3.5), (3.6) and (3.7) by using different integration steps. The results for the parameter $\Delta\omega$ are shown in Fig. 2. Curve 3 show the result obtained with the non-stiff algorithm (3.7), (3.8) with $\Delta\tau = 10^{-2}$ s, curves 1 and 2 show the results of conventional filtering algorithm (3.5), (3.6) with $\Delta\tau = 10^{-2}$ s and $\Delta\tau = 5 \cdot 10^{-3}$ s, respectively, and the dashed line is the exact solution.

It can be seen that, even with integration step $\Delta\tau = 10^{-2}$ s, the filtering algorithm gives acceptable accuracy $\varepsilon \leqslant 1\%$ with $S_0 = T/\Delta\tau = 200$ integration steps, whereas we must have $\Delta\tau = 10^{-3}$ s and $S_0 = 2000$ to achieve the same accuracy with the conventional algorithm.

Hence, the new method enables the step $\Delta\tau$ to be increased considerably while yielding guaranteed computational accuracy.

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